Assignment 2

Coverage: 15.2, 15.3 in Text.

Exercises: 15.2. no 23, 25, 27, 31, 35, 39, 55, 57, 61, 65, 69, 71, 75, 77, 79. 15.3. no 5, 7, 12, 15, 18, 29, 30.

Submit 15.2 no. 57, 61, 71; 15.3 no 15, 18 by Sept 21.

Supplementary Problems

1. Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(x) = 1$ for $x \in S$ and $\chi_S(x) = 0$ otherwise. Prove the following identities:

   (a) $\chi_{A\cup B} \le \chi_A + \chi_B$.

   (b) $\chi_{A\cup B} = \chi_A + \chi_B$ if and only if $A \bigcap B = \phi$, that is, A and B are disjoint.

   (c) $\chi_{A\cap B} = \chi_A \chi_B$.

2. Let f be integrable in a domain D which satisfies $A \le f \le B$ for two numbers A and B everywhere. Show that
$$A|D| \le \int_D f \le B|D| \ ,$$
   where $|D|$ is the "area" of D.

3. Show that a nonnegative, continuous function in a region has zero integral must be the zero function. Does it continue to hold without the continuity assumption?

On The Notion Of Area

Early from high school we learned the formulas for the area of common geometric figures such as rectangles, square, triangles and circles. We were asked to memorize these formula, but seldom was the notion of area explained. Indeed, whenever a geometric figure, especially those bounded by closed curves, is encountered, we have no doubt that it has an area. This faith carries from primary school through high school until we enter college.

Finally, we learned in first year calculus that how the area is defined. When the figure is the region enclosed by the graph of some nice, nonnegative function f, the x-axis, and vertical lines $x = a, x = b$, the area of this region is defined to be
$$A = \int_a^b f(x)\, dx \ .$$
For instance, taking $f(x) = \sqrt{a^2 - x^2}, x = 0, x = a$, we find a quarter of a disk with radius a is equal to $\pi a^2/4$, hence the area of the disk is the famous formula πa^2.

However, such definition is not general enough to embrace all figures. It only applies to a special class of figures as described above. We need to work harder for a more general definition. The resolution comes from double integral. Double integration of a function of two variables over a region is defined via Riemann sums. When the integrand is the constant 1, the Riemann sums are clearly approximate areas. Hence we can define the area of a figure (region) to be

$$A = \iint_{R_0} \chi_D(x,y)\, dA\ ,$$

where R_0 is a rectangle containing D. The area of D is well-defined when its characteristic function χ_D is integrable over R_0. (We have shown that the integral gives the same value over different R_0's.)

There is an interesting question concerning this definition of area. Namely, a Riemann sum is defined in terms of partitions where only subrectangles whose sides are parallel to the x-, y-axes are allowed. What happens when the subrectangles are not of this type? The question boils down to a simpler question: Can we use this definition to compute the area of a "tilted" rectangle?

When the boundary of the region D is given by one or several piecewise smooth, simple closed curves, we explained in class that χ_D is integrable. Since almost all figures we encounter are of this type, this definition is general enough to embrace almost all applications. However, mathematicians were not satisfied with this definition, thinking that it is still too narrow. They went on to study integrability more thoroughly, and the subject of measure theory were born. You will learn measure theory in Real Analysis.